Entropy production in ticking clocks: Fundamental limits of timekeeping

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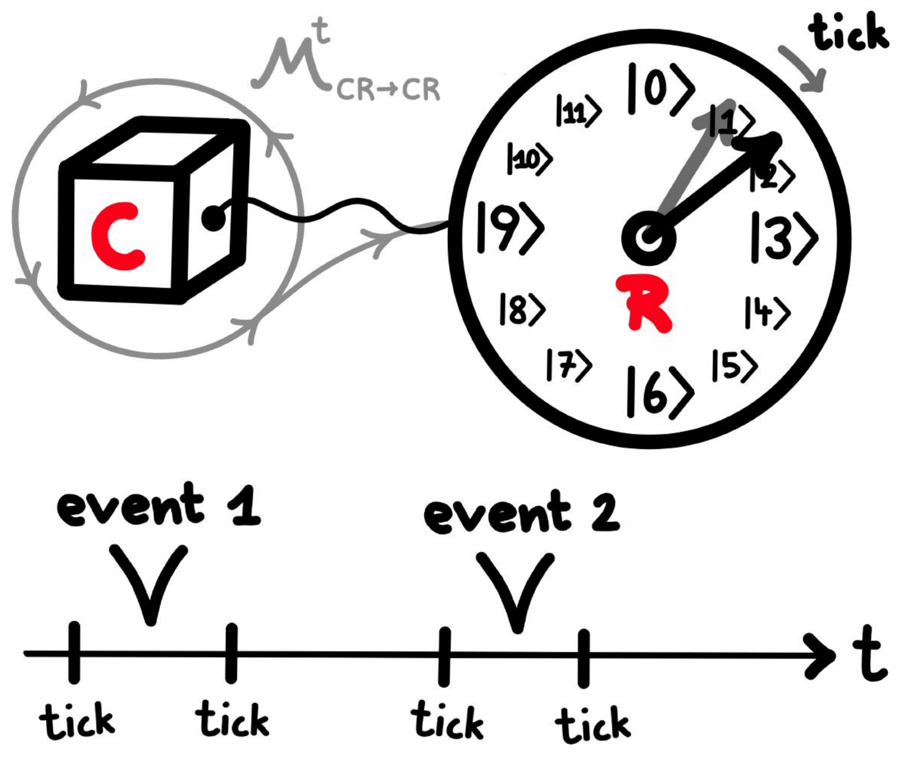
Ticking clock model

A ticking clock is a timekeeping device that **autonomously** outputs time information in the form of individual ticks

- modelled as a bipartite quantum system ρ_{CR} composed of clockwork C and register R living in $\mathcal{H}_{\mathrm{C}} \otimes \mathcal{H}_{\mathrm{R}}$ with $\dim(\mathcal{H}_{\mathrm{C}}) = d$ and $\dim(\mathcal{H}_{\mathrm{R}}) = N_{\mathrm{T}} + 1$
  - register states $\{|0\rangle_{\mathrm{R}}, |1\rangle_{\mathrm{R}}, \dots, |N_{\mathrm{T}}\rangle_{\mathrm{R}}\}$ represent no tick, 1 tick, …, and N_{T} ticks having occurred
  - $\rho_{\mathrm{CR}}(0) = \rho_{\mathrm{C}}^0 \otimes |0\rangle\langle 0|_{\mathrm{R}}$
  - $\rho_{\mathrm{CR}}(t) = \sum_{k=0}^{N_{\mathrm{T}}} \tilde{\rho}_{\mathrm{C}}^{(k)}(t) \otimes |k\rangle\langle k|_{\mathrm{R}}$

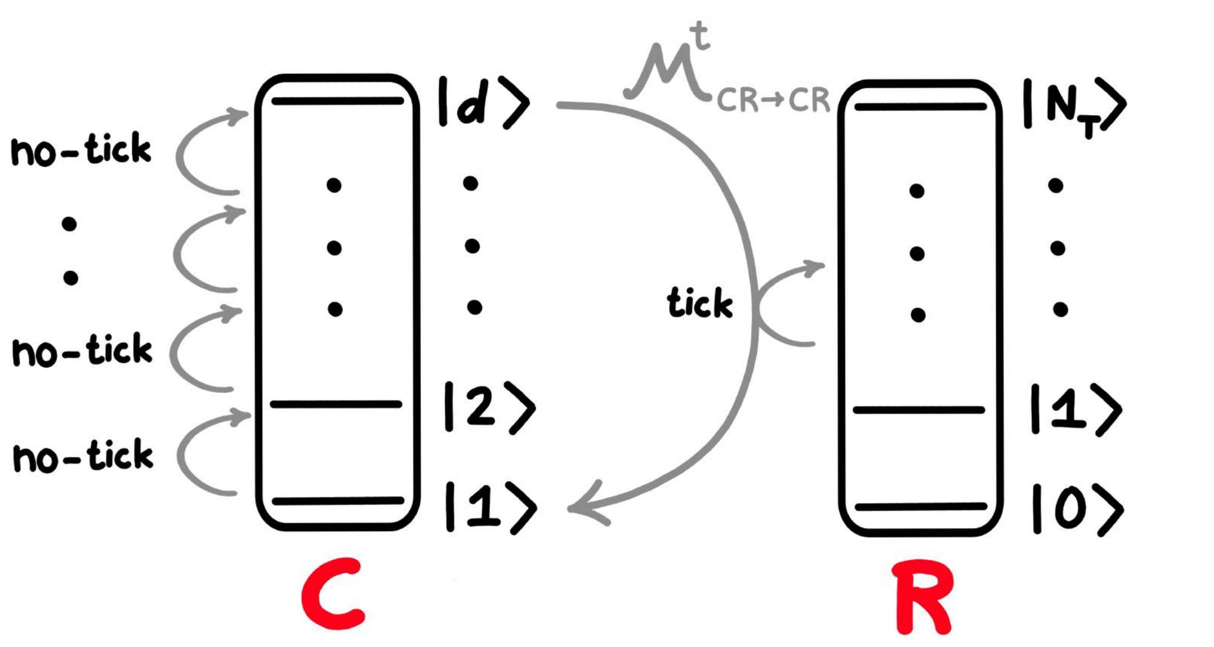


- dynamics are given by a family of CPTP maps $\mathcal{M}^t_{\mathrm{CR}\to\mathrm{CR}}(\cdot) = e^{\mathcal{L}_{\mathrm{CR}} t}(\cdot)$ parametrized by coordinate time $t \geq 0$
  - Lindbladian $\mathcal{L}_{\mathrm{CR}}(\cdot) = -i[\bar{H}, (\cdot)] + \sum_j \bar{L}_j(\cdot)\bar{L}_j^\dagger - \frac{1}{2}\{\bar{L}_j^\dagger \bar{L}_j, (\cdot)\} + \bar{J}_j(\cdot)\bar{J}_j^\dagger - \frac{1}{2}\{\bar{J}_j^\dagger \bar{J}_j, (\cdot)\}$, where $\bar{H} = H \otimes \mathbb{1}_{\mathrm{R}}$, $\bar{L}_j = L_j \otimes \mathbb{1}_{\mathrm{R}}$, $\bar{J}_j = J_j \otimes O_{\mathrm{R}}$ with $O_{\mathrm{R}} = |1\rangle\langle 0|_{\mathrm{R}} + |2\rangle\langle 1|_{\mathrm{R}} + \dots + |N_{\mathrm{T}}\rangle\langle N_{\mathrm{T}} - 1|_{\mathrm{R}}$

Example: Ladder ticking clock

Choosing some orthonormal basis $\{|j\rangle_{\mathrm{C}}\}_{j=1}^d$ of $\mathcal{H}_{\mathrm{C}}$, the clock is given by

- $L_j = \begin{cases} |j+1\rangle\langle j|_{\mathrm{C}}, & \text{for } j \neq d \\ 0, & \text{for } j = d \end{cases}$
- $J_j = \begin{cases} 0, & \text{for } j \neq d \\ |1\rangle\langle d|_{\mathrm{C}}, & \text{for } j = d \end{cases}$
- $H = 0$ and $\rho_{\mathrm{C}}^0 = |1\rangle\langle 1|_{\mathrm{C}}$



Most accurate classical clock achieving $R_k = kR_1$ with $R_1 = d$ via reset

- classical ⇔ clockwork remains incoherent $\rho_{\mathrm{C}}(t) = \sum_j p_j(t)|j\rangle\langle j|_{\mathrm{C}}$
- reset ⇔ $\mathcal{C}_{\mathrm{tick}}(\rho_{\mathrm{C}}) = \rho_{\mathrm{C}}^0 \;\forall \rho_{\mathrm{C}}$, where $\mathcal{C}_{\mathrm{tick}}(\cdot) = \sum_j J_j (\cdot) J_j^\dagger$

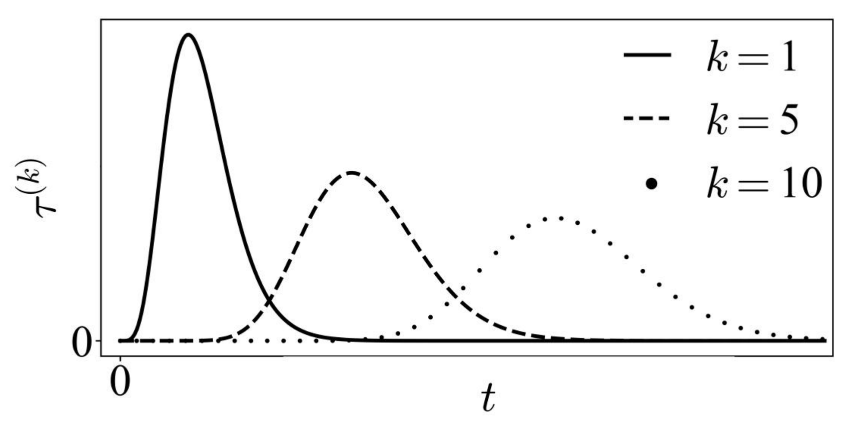
Quantum reset clocks can achieve a higher accuracy $R_k = kR_1$ with $R_1 \approx d^2$

Measure of accuracy

based on delay functions $\{\tau^{(k)}(t)\}_{k=1}^{N_{\mathrm{T}}}$

- probability that $(k-1)$ ticks occur during $[0, t)$ and kth tick occurs during $[t, t+\delta t]$ is $\delta t \cdot \tau^{(k)}(t)$ with $\delta t > 0$
- accuracy of kth tick is $R_k = \mu_k^2/\sigma_k^2$

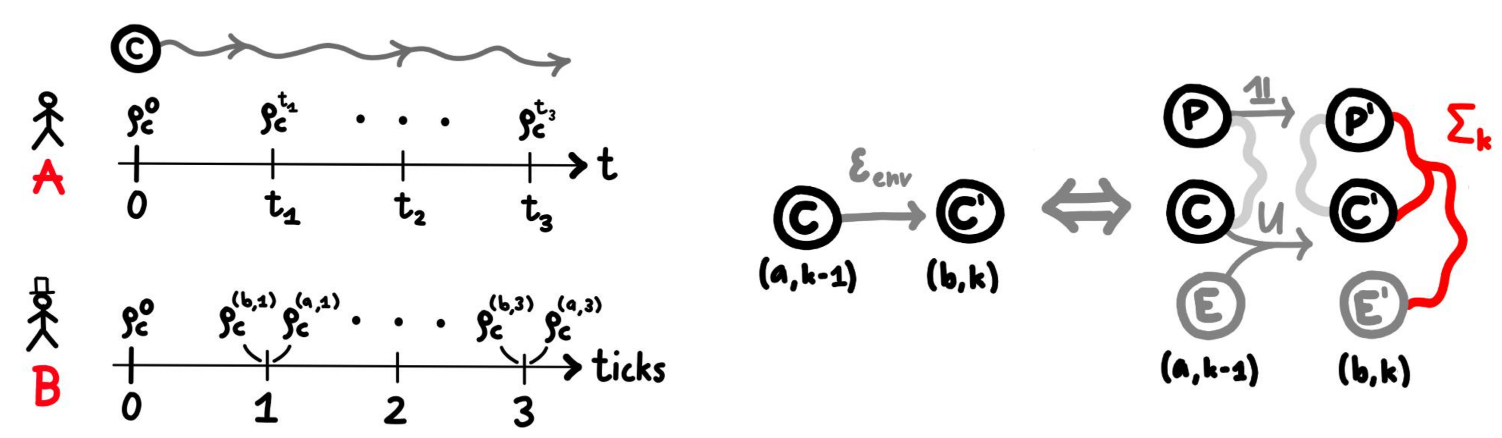
For ladder ticking clock:



Measure of entropy production

$\Sigma_k = S(\rho_{\mathrm{CP}}^{(\mathrm{b},k)})$ quantifies exchanged information between ticking clock and its environment during the kth tick

- entropy is **observer-dependent** ⇒ relevant observer (B) only has knowledge about ticks (not coordinate time t itself)



Relationship between accuracy and entropy production

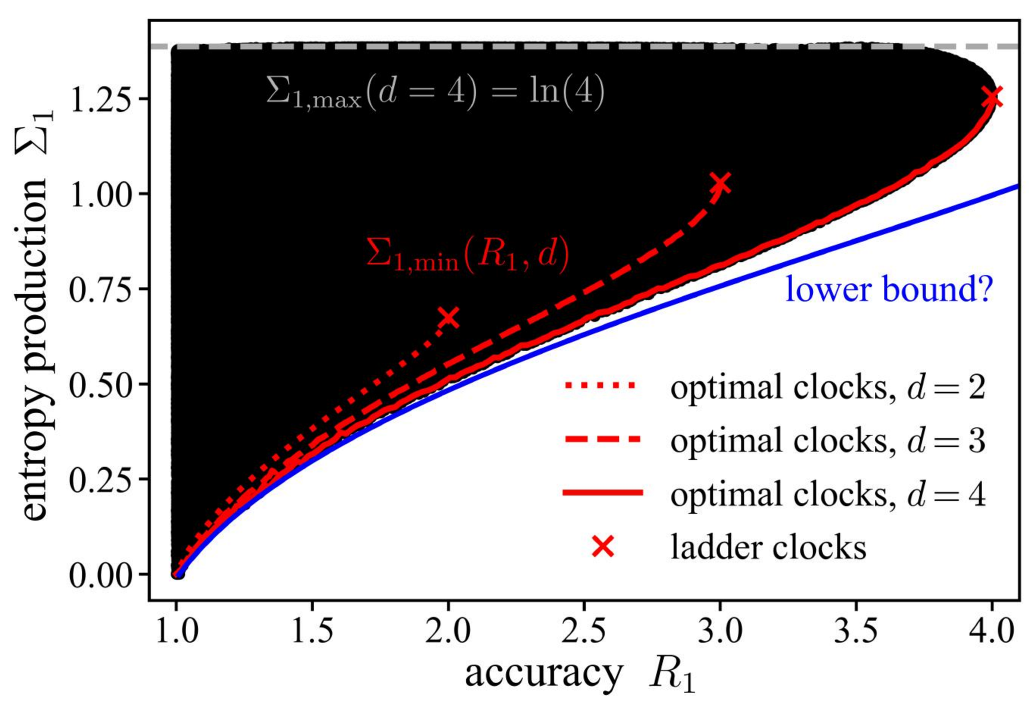
Vanishing entropy production per tick can be achieved by

- classical reset clock exhibiting Poissonian tick statistics with $R_1 = 1$
  - $H = 0,\ L_j = 0\ \forall j,\ J = \sqrt{\lambda}\mathbb{1}$, and $\rho_{\mathrm{C}}^0 = |\Psi\rangle\langle\Psi|_{\mathrm{C}}$
- "quasi-ideal" quantum reset clock with $R_1 \approx d^2 \to \infty$
  - $H \neq 0,\ L_j = 0\ \forall j,\ J_j \neq 0\ \forall j$, and $\rho_{\mathrm{C}}^0 = |\Psi\rangle\langle\Psi|_{\mathrm{C}}$

⇒ proof crucially relies on vanishing no-tick generators $\{L_j\}$

Note: Classical clocks with $L_j = 0\ \forall j$ achieve maximal accuracy of $R_1 = 1$

Every classical ticking clock must produce a minimal amount of entropy per tick which increases with increasing accuracy



References

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