Sum-of-Squares Bounds for Quantum Optimal Control

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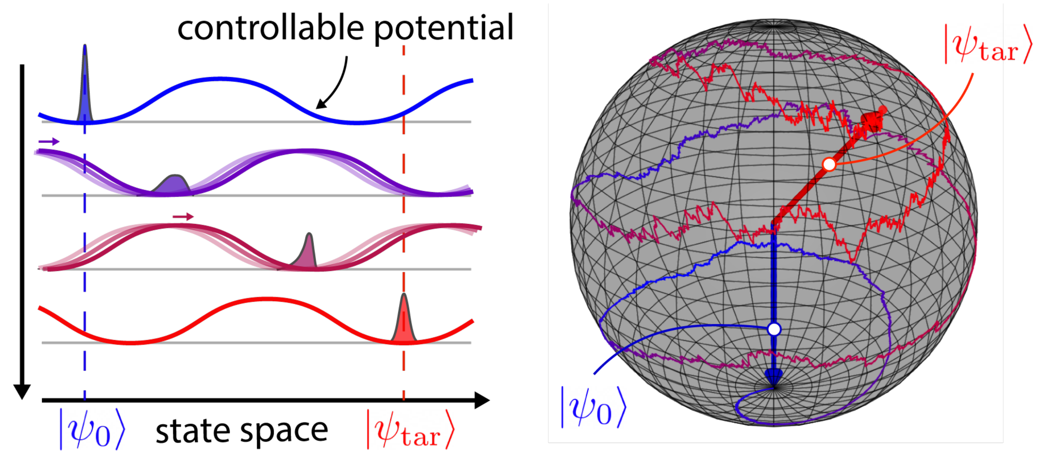
Relevance

The speed at which quantum states can evolve and the accuracy with which they can be prepared in the presence of noise put a limit on the capabilities of quantum information technology. Quantifying this limit is thus crucial for the design of quantum devices and the evaluation of their potential.



Example – Quantum Brachistochrone Problem

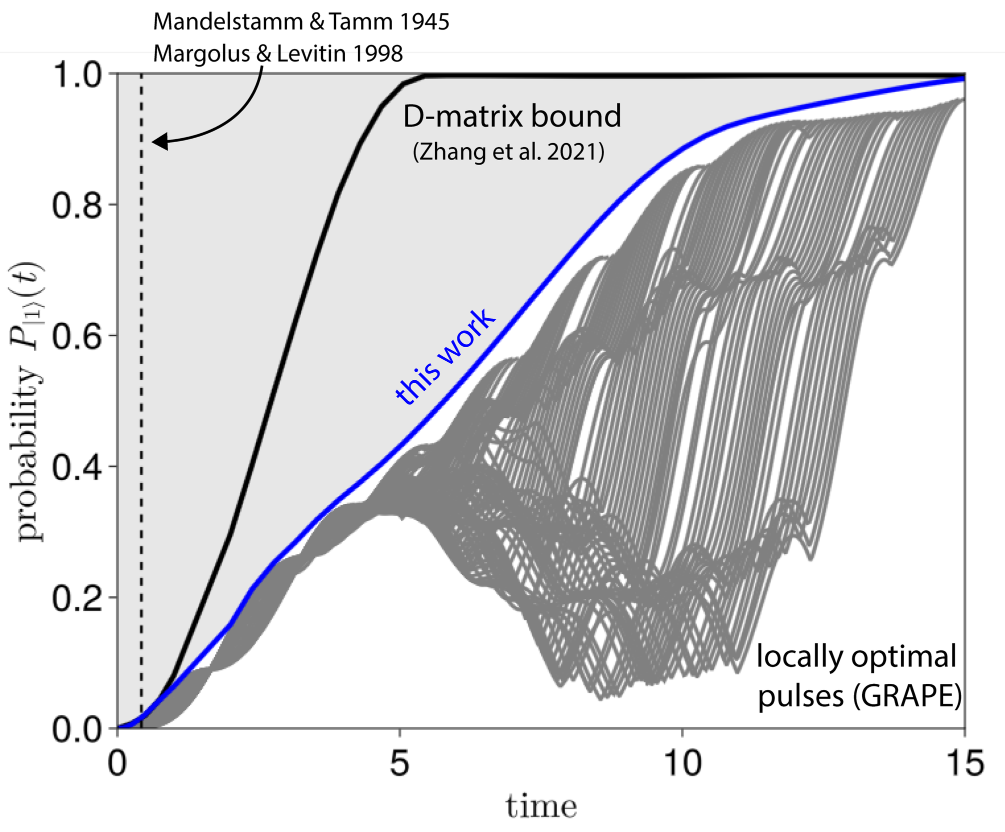
Given a quantum system, what is the minimal time to evolve its ground state $|\psi_0\rangle$ to a target state $|\psi_{\mathrm{tar}}\rangle$ by way of controlling the system's Hamiltonian?

Open-loop quantum brachistochrone problems

Closed transmon qubit system [Zhang et al. 2021]:

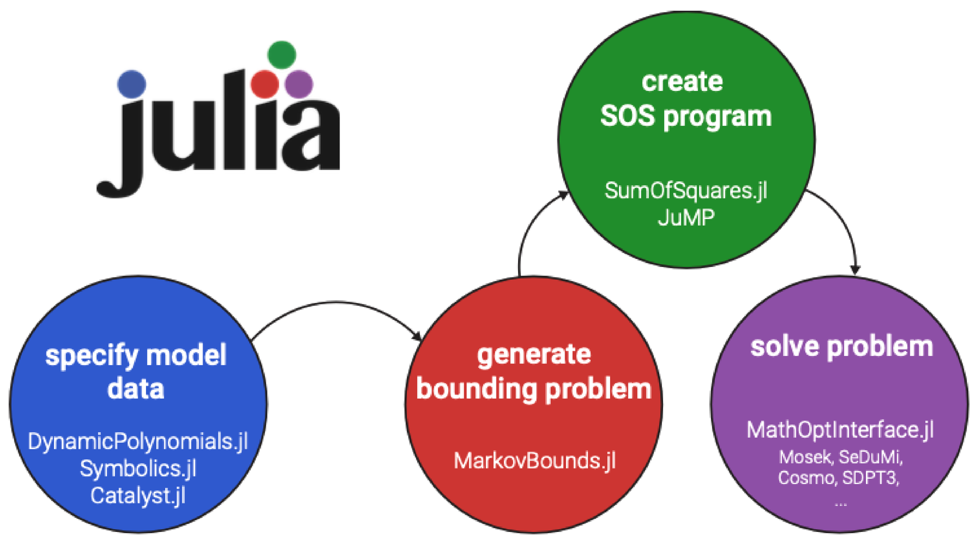
$$H(u) = \begin{bmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{bmatrix} + u \begin{bmatrix} 0 & \mu_{01} & 0 \\ \mu_{10} & 0 & \mu_{12} \\ 0 & \mu_{21} & 0 \end{bmatrix}, \quad u \in [-1, 1]$$

- global bounds complement local gradient-based optimization methods
- the proposed SoS framework improves upon a range of quantum speed limits by accounting for technological constraints and detailed system information



MarkovBounds.jl

Julia's optimization ecosystem enables simple use



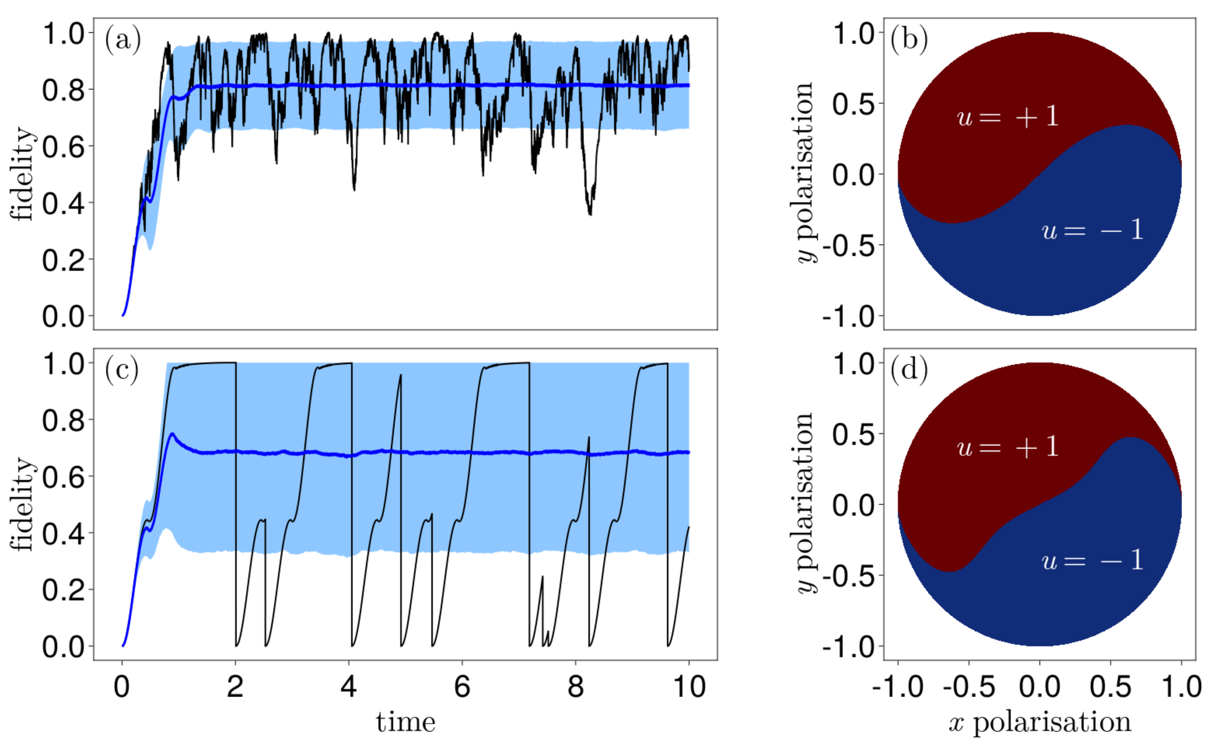
GitHub.com/FHoltorf/MarkovBounds.jl

Closed-loop control of continuously observed qubit

Single qubit with σ_- measurement:

$$H(u) = \frac{\Delta}{2}\sigma_z + u\sigma_x, \quad u \in [-1, 1]$$

- approximate value function enables construction of certifiably near-optimal controllers
- computational proof of superiority of homodyne detection over photon counting



Homodyne detection

Degree d	Fidelity bound	Comp. time [s]
2	0.8502	0.008
4	0.8111	0.078
6	0.7973	0.64
8	0.7893	5.0
10	0.7856	27.9
Best known fidelity: 0.7750		

Photon counting

Degree d	Fidelity bound	Comp. time [s]
2	0.9602	0.0043
4	0.7497	0.031
6	0.7153	0.180
8	0.6902	1.67
10	0.6798	14.9
Best known fidelity: 0.6547		

General framework

Quantum optimal control

Suppose a quantum system is described by a Hamiltonian,

$$H(u) = H_0 + \sum_{k=1}^{K} u_k H_k\,,$$

where H_k are the control fields with control drives $u = [u_1 \ \cdots \ u_K] \in U = [-1,1]^K$.

Under continuous observation, the jump-diffusion dynamics of the system state are governed by the quantum filtering equation

$$d\rho_t = \mathcal{L}(H(u_t))\rho_t\, dt + \mathcal{G}\rho_t\, d\xi_t. \qquad \text{(QFE)}$$

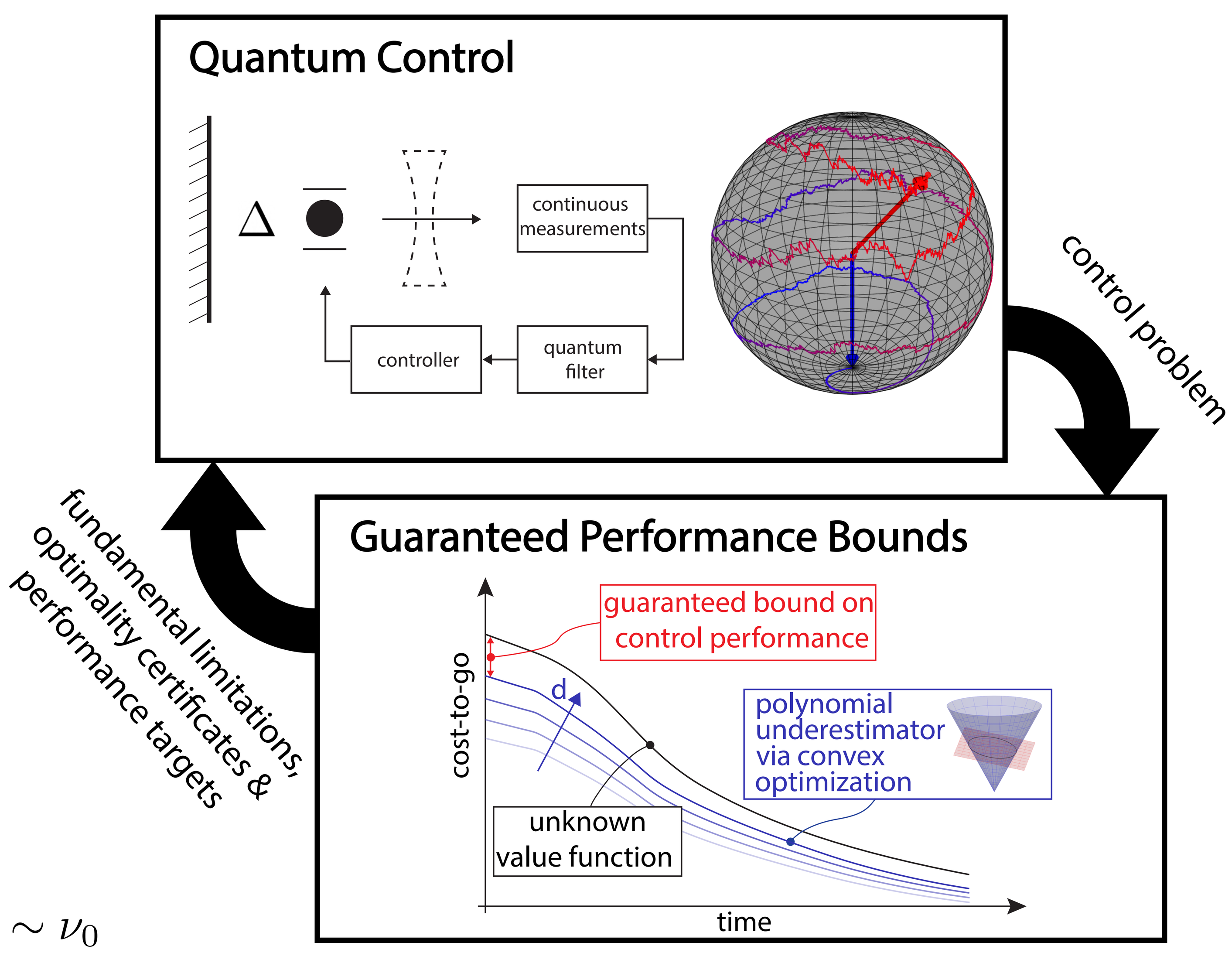
Goal

Bound the optimal performance J^* attainable by the *best possible controller*

$$J^* = \inf_{u_t \in U} \mathbb{E}\left[\int_0^T \ell(\rho_t, u_t)\, dt + m(\rho_T)\right]$$

s.t. ρ_t satisfies (QFE) on $[0,T]$ with $\rho_0 \sim \nu_0$
u_t is an admissible controller.

IDEA: FIND MAXIMAL POLYNOMIAL HAMILTON-JACOBI-BELLMAN SUBSOLUTION VIA SUM-OF-SQUARES PROGRAMMING



Dynamic programming & SoS

The optimal value function w of the quantum optimal control problem is characterized as the maximal Hamilton-Jacobi-Bellman subsolution [Bhatt and Borkar, 1996]

$$J^* = \sup_{w \in \mathcal{C}^2([0,T]\times B)} \int_B w(0,\,\cdot\,)\, d\nu_0$$

s.t. $\mathcal{A}w + \ell \geq 0$ on $[0,T] \times B \times U$
$w(T,\,\cdot\,) \leq m$ on B

- SoS techniques [Lasserre, 2010] yield a hierarchy of tractable convex (SDP) restrictions by restricting w to be a polynomial of degree at most d.
- SDP restrictions furnish monotonically improving lower bounds for J^* as d increases; convergence is guaranteed if the value function is sufficiently smooth.
- polynomial underapproximators to the value function are a byproduct and can inform controller design.

References

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